COMBINATORICA
Akadémiai Kiadó — Springer-Verlag

COMBINATORICA 9 (4) (1989) 321—337

THE PERFECTLY MATCHABLE SUBGRAPH POLYTOPE OF AN ARBITRARY GRAPH

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Received October 20, 1987
Revised June 18, 1988

The Perfectly Matchable Subgraph Polytope of a graph $G=(V,E)$, denoted by $PMS(G)$, is the convex hull of the incidence vectors of those $X\subseteq V$ which induce a subgraph having a perfect matching. We describe a linear system whose solution set is $PMS(G)$, for a general (nonbipartite) graph G. We show how it can be derived via a projection technique from Edmonds' characterization of the matching polytope of G. We also show that this system can be deduced from the earlier bipartite case [2], by using the Edmonds—Gallai structure theorem. Finally, we characterize which inequalities are facet inducing for $PMS(G)$, and hence essential.

1. Introduction

A *matching* in a graph $G=(V,E)$ is a set M of edges such that each node is incident with at most one member of M. Those nodes incident with members of M are said to be *saturated* by M. If all nodes are saturated by M, then M is a *perfect matching*. We say that $S\subseteq V$ induces a *perfectly matchable subgraph* of G if the subgraph $G[S]$ induced by S has a perfect matching. We let $\mathcal{W}$ be the set of all such subsets of V, and adopt the convention that $\emptyset\in\mathcal{W}$, i.e. the empty subgraph is perfectly matchable. The *perfectly matchable subgraph polytope* of G, denoted by $PMS(G)$, is the convex hull of the $0-1$ incidence vectors of the members of $\mathcal{W}$.

In [2] we gave a set of inequalities sufficient to define $PMS(G)$ for a bipartite graph G. In this paper we give such a system for the general case when G may be nonbipartite. We first show how a projection method described in [2] can be used to obtain this system. We then describe how the earlier bipartite result, together with the Edmonds—Gallai structure theorem can be used to give a proof.

Optimizing over $PMS(G)$ can be accomplished by solving a special case of the weighted matching problem. For suppose we have a vector $c=(c_v\colon v\in V)$ of real node weights and we wish to find $x\in PMS(G)$ which maximizes cx, or equivalently, $S\in\mathcal{W}$ for which $\sum(c_v\colon v\in S)$ is maximum. We define $\tilde{c}_{uv}=c_u+c_v$ for every edge $uv\in E$ and then find a (not necessarily perfect) matching M of G for which $\sum(\tilde{c}_e\colon e\in M)$ is maximized. The nodes saturated by M provide the solution. In fact, this relationship provides the basis for a derivation of the linear description of $PMS(G)$.

In the next section, we describe the projection method, based on Benders' decomposition, which we introduced in [2]. We also show how it applies to $PMS(G)$ for a general graph G. In Section 3 we discuss the relationship of the bipartite and nonbipartite theorems. In particular, we give a second derivation of the general

AMS subject classification (1980): 52A25, 05C70

result, from the bipartite result, plus the Edmonds—Gallai structure theorem. In Section 4 we characterize the facet inducing inequalities for $PMS(G)$, which enables us to give a minimal defining linear system. Then in Section 5 we present some concluding remarks.

2. Projection and Cones

First we describe a general projection method. Suppose we are given a polyhedron

$$\begin{aligned} Z = \{(u, x)\colon\ & A^1u+B^1x = b^1,\\ & A^2u+B^2x \leqq b^2\\ & u \geqq 0,\ x \in D\} \end{aligned}$$

where A^1, A^2, B^1, B^2 are matrices, b^1, b^2 are vectors and D is a set to which all feasible x belong. Let X denote the *projection* of Z onto the subspace of x variables, that is,

$$X = \{x\colon \text{there exists } u \text{ such that } (u, x) \in Z\}.$$

We wish to obtain a linear system whose solution set is X.

We define the cone

$$W = \{(y, z)\colon yA^1+zA^2 \geqq 0,\ z \geqq 0\}.$$

Let $\hat{W}$ be any (finite) set of generators of W. That is, we should have $(y, z) \in W$ if and only if (y, z) can be expressed as a nonnegative linear combination of members of $\hat{W}$.

Then

$$X = \{x \in D\colon (yB^1+zB^2)x \leqq yb^1+zb^2 \text{ for all } (y, z) \in \hat{W}\}. \tag{2.1}$$

In fact (2.1) is quite easy to prove. First suppose y, z satisfy $yA^1+zA^2 \geqq 0$, $z \geqq 0$ and let $(u, x) \in Z$. Then

$$yB^1x+zB^2x \leqq yb^1+zb^2-(yA^1+zA^2)u \leqq yb^1+zb^2$$

and hence x satisfies the linear system of (2.1). Conversely, suppose $x \notin X$, i.e., there exists no $u \geqq 0$ such that

$$A^1u = b^1-B^1x$$

$$A^2u \leqq b^2-B^2x.$$

Then by Farkas' lemma there exist y and z satisfying

$$yA^1+zA^2 \geqq 0$$

$$z \geqq 0$$

$$y(b^1-B^1x)+z(b^2-B^2x) < 0. \tag{2.2}$$

But then $(y, z) \in W$ and so there must be some member $(\hat{y}, \hat{z})$ of $\hat{W}$ which also satisfies (2.2), i.e, $(\hat{y}B^1+\hat{z}B^2)x > \hat{y}b^1+\hat{z}b^2$. Therefore x does not satisfy the system (2.1).

In general, the main problem we have to solve is the following: Given a cone $W = \{(y, z)\colon yA^1+zA^2 \geqq 0,\ z \geqq 0\}$, find a finite set $\hat{W}$ of generators. Such a set can be characterized as follows: First, let I be an index set for the inequalities defining W. Let

$$W^{=} = \{(y, z): \; yA^1 + zA^2 = 0, \; z = 0\}.$$

The set $W^{=}$ is called the *lineality space* of W and consists of all those $w \in W$ for which $\alpha w \in W$ for all $\alpha \in \mathbf{R}$. Let $\hat{W}^{=}$ be any basis of $W^{=}$. (Note that if $W^{=}$ consists of just the zero vector, then $\hat{W}^{=} = \emptyset$.)

For any $(\tilde{y}, \tilde{z}) \in W$ we let $I^{=}(\tilde{y}, \tilde{z})$ be the set of indices in I for which the corresponding inequalities hold as equations for $(\tilde{y}, \tilde{z})$. (Then $(\tilde{y}, \tilde{z}) \in W^{=}$ if and only if $I^{=}(\tilde{y}, \tilde{z}) = I$.) Let $\mathscr{R}$ be the set of all maximal proper subsets J of I such that $J = I^{=}(\tilde{y}, \tilde{z})$ for some $(\tilde{y}, \tilde{z}) \in W$. Then for any $J \in \mathscr{R}$, let $r(J)$ consist of all those $(y, z) \in W$ for which $I^{=}(y, z) = J$. The *extreme* elements of W are the members of $r(J)$, for any $J \in \mathscr{R}$. Let $\hat{W}^{+}$ consist of one nonzero member of $r(J)$ for each $J \in \mathscr{R}$. Then every member of W can be expressed as a linear combination of members of $\hat{W}^{=}$ plus a nonnegative linear combination of members of $\hat{W}^{+}$. Thus if we let $\hat{W} = \hat{W}^{=} \cup (-\hat{W}^{=}) \cup \hat{W}^{+}$ we have a set of generators as required.

If $W^{=}$ contains only the zero vector, then W is a *pointed* cone. In this case the sets $r(J)$ each consist of all positive multiples of a single member of W. These sets are called the *extreme rays* of the cone W. (This is the case we encounter here for nonbipartite graphs.)

Now we describe how projection can be used to obtain $PMS(G)$ for a graph $G = (V, E)$. For any $S \subseteq V$ we let $\delta(S)$ denote the coboundary of S, i.e., the set of edges with exactly one end in S. We write $\delta(v)$ for $\delta(\{v\})$, for any $v \in V$. We let $\gamma(S)$ denote the set of edges having both ends in S. For any finite set J and real vector $(x_j: j \in J)$ and $I \subseteq J$ we let $x(I)$ denote $\sum (x_j: j \in I)$.

The *matching polytope* of G, denoted by $M(G)$, is the convex hull of the incidence vectors of the (not necessarily perfect) matchings of G. The following gives a linear system sufficient to define $M(G)$.

Theorem 2.1. (Edmonds [5]). *For any graph* $G = (V, E)$, $M(G) = \{u \in \mathbf{R}^E$:

(2.4) $$u \geqq 0,$$

(2.5) $$u(\delta(v)) \leqq 1 \quad \textit{for all} \quad v \in V,$$

(2.6) $$u(\gamma(S)) \leqq (|S| - 1)/2 \quad \textit{for all} \quad S \in \mathscr{Q}\}$$

where $\mathscr{Q} = \{S \subseteq V: \; |S| \geqq 3, \text{ odd}\}$.

If G is bipartite, then the inequalities (2.6) can be omitted, and the result is equivalent to the Birkhoff—von Neumann theorem which asserts that a doubly stochastic matrix is a convex combination of permutation matrices. In this paper, our main subject of interest is the case of nonbipartite graphs. However, most of the development remains valid, and considerably simpler, for bipartite graphs, when we take $\mathscr{Q} = \emptyset$. Generally we will omit pointing this out, however we will indicate when differences arise.

Suppose we add a slack variable x_v' to each inequality (2.5) and then make the substitution $x_v = 1 - x_v'$. Then we obtain the following:

Corollary 2.2. *The polyhedron Z defined by the following linear system has only integer vertices:*

$$u \geqq 0, \quad 0 \leqq x \leqq 1;$$
$$u(\delta(v)) - x_v = 0 \quad \text{for all} \quad v \in V;$$
$$u(\gamma(S)) \leqq (|S|-1)/2 \quad \text{for all} \quad S \in \mathcal{Q}.$$

In fact, each vertex (u, x) of Z satisfies the following: u is the incidence vector of a matching of G and x is the incidence vector of the vertices saturated by the matching. Conversely, each such u, x defines a vertex of Z. Therefore $PMS(G)$ is simply the projection of Z onto the subspace of the x variables.

In order to apply the projection method of this section, we first identify the various components of our linear system:

A^1 is the node-edge incidence matrix of G;
B^1 is the negative of an identity matrix;
b^1 is a zero vector;
A^2 has one row for each $S \in \mathcal{Q}$, and that row is the incidence vector of $\gamma(S)$;
B^2 is zero;
b^2 has one entry for each $S \in \mathcal{Q}$ having the value $(|S|-1)/2$.

Finally, $D = \{x\colon 0 \leqq x \leqq 1\}$.

Our main object of attention is the cone $W = \{(y, z)\colon yA^1 + zA^2 \geqq 0, \ z \geqq 0\}$. That is, we assign a value y_i to each $i \in V$ and a nonnegative value z_S to each $S \in \mathcal{Q}$ such that

$$y_i + y_j + \sum (z_S\colon i, j \in S, \ S \in \mathcal{Q}) \geqq 0 \quad \text{for all} \ ij \in E.$$

Proposition 2.3. *W is a pointed cone if and only if every component of G is nonbipartite.*

Proof. $W^=$ is the set of all vectors of the form $(y, 0)$ where y satisfies $y_i + y_j = 0$ for all $ij \in E$. If a component has an odd cycle, then these equations imply $y_i = 0$ for all nodes i of this cycle, which in turn implies that $y_i = 0$ for all i in the nodeset of the component. If a component is bipartite, with bipartition of the nodeset $K_1 \cup K_2$, then the vector $\hat{y}$ defined by

$$\hat{y}_i = \begin{cases} \alpha & \text{for} \quad i \in K_1 \\ -\alpha & \text{for} \quad i \in K_2 \\ 0 & \text{for} \quad i \notin K_1 \cup K_2 \end{cases}$$

is in the cone for all α. Therefore $W^= = \{0\}$ if and only if every component of G is nonbipartite. ∎

If G is not connected, then a linear system sufficient to define $PMS(G)$ is obtained by concatenating such systems for the various components. Hence we can assume that G is connected. In this case, in principle, all we have to do is give a complete set of generators of W. If G is nonbipartite, this is equivalent to describing the extreme rays of W. However, in fact we can do less than that for there will be extreme rays of W which do not yield facet inducing (essential) inequalities for $PMS(G)$. Moreover, there will be distinct extreme rays which yield the same facet inducing inequality for $PMS(G)$.

Proposition 2.4. *If $G = (V, E)$ is connected and nonbipartite, then $PMS(G)$ is of full dimension.*

Proof. We exhibit $|V|+1$ members of $PMS(G)$ which are affinely independent. Let T be a spanning tree of G and let j be an edge which creates an odd cycle when added to T. For each $k\in E(T)\cup\{j\}$ we define a vector $x^k\in PMS(G)$ by letting

$$x_v^k = \begin{cases} 1 & \text{if } v\in V \text{ is an end of } k \\ 0 & \text{if } v\in V \text{ is not incident with } k. \end{cases}$$

An easy inductive argument shows that these vectors are linearly independent. Moreover, $x^k(V)=2$ for all $k\in E(T)\cup\{j\}$. Hence the zero vector, which is also in $PMS(G)$, cannot be expressed as an affine combination of these vectors, so these give the required set of $|V|+1$ vectors. ▌

A consequence of Proposition 2.4 is that when G is nonbipartite and connected, the minimal defining linear system for $PMS(G)$ is unique, up to positive multiples of the inequalities. We say that two valid inequalities for $PMS(G)$ are *equivalent* if one is a positive multiple of the other. We already have the inequalities $0\leqq x_v\leqq 1$ in our defining system for $PMS(G)$; they made up the definition of D. We say that a valid inequality is *trivial* if it is a positive multiple of one of these inequalities. Otherwise, we say that it is *nontrivial*. Similarly, we call a facet of $PMS(G)$ *trivial* if it is generated by a trivial inequality and otherwise *nontrivial.*

In [2] we showed that if $G=(V_1\cup V_2, E)$ is bipartite and connected, then $PMS(G)$ is of dimension $|V_1\cup V_2|-1$. The unique (up to positive multiples) equation satisfied by all members of $PMS(G)$ is $x(V_1)-x(V_2)=0$. In this case two valid inequalities for $PMS(G)$ are equivalent if one is obtained from the other by multiplying by a positive constant and then adding an arbitrary multiple of the equation $x(V_1)-x(V_2)=0$. Again, trivial inequalities are those equivalent to an inequality $x_v\geqq 0$ or $x_v\leqq 1$, for some $v\in V_1\cup V_2$.

Proposition 2.5. *For any $(y,z)\in W$, the inequality $ax\leqq a_0$ is valid for $PMS(G)$, where a and a_0 are defined by*

$$\begin{aligned} a &= -y \\ a_0 &= \sum (z_S\cdot(|S|-1)/2\colon\ S\in\mathcal{Q}). \end{aligned} \tag{2.7}$$

Conversely, if $ax\leqq a_0$ is a nontrivial facet inducing inequality for $PMS(G)$, then there exists an extreme $(y,z)\in W$ satisfying (2.7).

Proof. Apply formulae (2.1) to the matrices B^1, B^2 and vectors b^1, b^2 defined above. ▌

Note that there may be many extreme members of W (all having the same y-component) which satisfy (2.7). They will all yield the same valid inequality for $PMS(G)$. What is important for us is the fact that the lefthand side of a facet inducing inequality depends only on y and the righthand side depends only on z.

We now describe a particular set of vectors of W which we will then show is sufficient to generate all nontrivial facets of $PMS(G)$. Let

$$\mathcal{T} = \{X\subseteqq V\colon \text{each component of } G|X| \text{ has an odd numbers of nodes}\}.$$

For any $A\subseteqq V$, we let $\Gamma(A)$ denote the neighbour set of A. That is, $\Gamma(A)$ consists of those nodes not in A but adjacent to at least one member of A. For any $X\in\mathcal{T}$

and any $\alpha>0$ we define the following vectors:

$$y_v^{X,\alpha}=\begin{cases}-\alpha & \text{if } v\in X\\ \alpha & \text{if } v\in\Gamma(X)\\ 0 & \text{otherwise,}\end{cases}$$

$$z_S^{X,\alpha}=\begin{cases}2\alpha & \text{if } S\in\mathcal{Q} \text{ and } G[S] \text{ is a component of } G[X],\\ 0 & \text{otherwise.}\end{cases}$$

Note that, for $X\in\mathcal{T}$, there may be singleton components of $G[X]$. However, $z_S^{X,\alpha}>0$ only if $|S|$ is odd and at least 3.

It is easy to verify that, for any $X\in\mathcal{T}$ and any $\alpha\geqq 0$, the vector $(y^{X,\alpha}, z^{X,\alpha})\in W$. We now show that if $ax\leqq a_0$ is a nontrivial facet inducing inequality for $PMS(G)$, then there exists $X\in\mathcal{T}$ and $\alpha>0$ such that $(y^{X,\alpha}, z^{X,\alpha})$ gives this inequality. Our proof makes use of the following two notions. A family $\mathcal{F}$ of subsets of V is said to be *nested* if, for any $S, T\in\mathcal{F}$, whenever $S\cap T\neq\emptyset$, either $S\subseteq T$ or $T\subseteq S$. If $\mathcal{F}$ is a nested family of sets, then we let $G\times\mathcal{F}$ denote the graph obtained from G by shrinking the maximal members of $\mathcal{F}$ to form *pseudonodes*. For any $S\in\mathcal{F}$, we let $\mathcal{F}[S]$ denote the subfamily of $\mathcal{F}$ consisting of all members of $\mathcal{F}$ properly contained in S. Thus $G[S]\times\mathcal{F}[S]$ is the graph obtained from $G[S]$, the subgraph of G induced by S, by shrinking all maximal members of $\mathcal{F}$ properly contained in S. See Figure 1. Note that $G[S]\times\mathcal{F}[S]$ can have multiple edges, whether or not G has multiple edges. However, shrinking cannot create loops, as such edges disappear in the shrinking process.

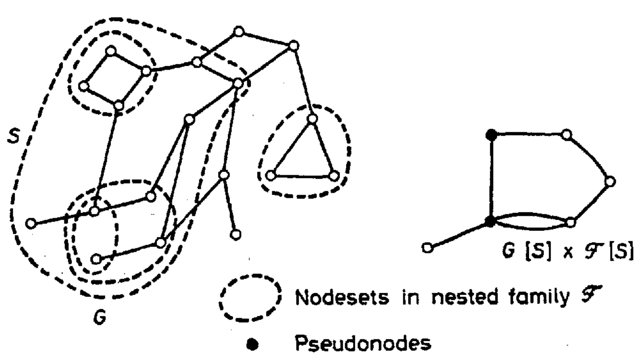

Fig. 1

Theorem 2.6. *Let $ax\leqq a_0$ be a nontrivial facet inducing inequality for $PMS(G)$, for a connected graph G. Then there exists $X\in\mathcal{T}$ and $\alpha>0$ such that*

$$a_v=-y_v^{X,\alpha} \text{ for all } v\in V,$$

$$a_0=\sum\left(z_S^{X,\alpha}\cdot(|S|-1)/2\colon S\in\mathcal{Q}\right).$$

Moreover, for each $S\in\mathcal{Q}$ such that $z_S^{X,\alpha}>0$, $G[S]$ is connected and non-bipartite.

Proof. By Proposition 2.5, there exists extreme $(y, z)\in W$ such that a and a_0 are given by (2.7). For any $z=(z_S\colon S\in\mathcal{Q})$ we let $Z(z)=\sum(z_S\cdot(|S|-1)/2\colon S\in\mathcal{Q})$. We establish four claims:

Claim 1. *For any $(y', z')\in W$ satisfying $y'\geqq y$, we must have $Z(z)-Z(z')\leqq y'(V)-y(V)$. If $y'\neq y$, then this inequality is strict.*

For let $a'x\leqq a_0'$ be the valid inequality for $PMS(G)$ corresponding to (y', z'), defined by (2.7). If for each $v\in V$, we add $(y_v'-y_v)$ times the inequality $x_v\leqq 1$, we obtain $ax\leqq a_0'+(y'(V)-y(V))$. Since $ax\leqq a_0$ is facet inducing, we must have $a_v'+(y'(V)-y(V))\geqq a_0$. If $y'\neq y$, then since $ax\leqq a_0$ induces a nontrivial facet, and we have obtained it from another inequality by adding a positive multiple of $x_v\leqq 1$ for at least one $v\in V$, we must have $a_v'+(y'(V)-y(V))>a_0$. For otherwise we would have expressed a facet inducing inequality as a nonnegative combination of other valid, nonequivalent inequalities.

Claim 2. *We can assume that $\mathcal{Q}'=\{S\in\mathcal{Q}\colon z_S>0\}$ is a nested family.*

For suppose $(y, z)\in W$ satisfying (2.7) is chosen such that $\sum(z_S\cdot|S|^2\colon S\in\mathcal{Q})$ is maximized. (Since $Z(z)=a_0$, and $z\geqq 0$, this maximum exists.) Suppose there exists $S, T\in\mathcal{Q}'$ such that $S\cap T\neq\emptyset$ but $S\nsubseteq T$ and $T\nsubseteq S$. Assume $z_S\leqq z_T$.

First suppose $|S\cap T|$ is even. Define y', z' as follows:

$$y_v'=\begin{cases} y_v & \text{if } v\notin S\cap T\\ y_v+z_S & \text{if } v\in S\cap T;\end{cases}$$

$$z_W'=\begin{cases} z_W+z_S & \text{if } W=S\setminus T \text{ or } W=T\setminus S \text{ and } |W|\geqq 3,\\ z_T-z_S & \text{if } W=T\\ 0 & \text{if } W=S\\ z_W & \text{if } W\in\mathcal{Q}\setminus\{S, T, S\setminus T, T\setminus S\}.\end{cases}$$

For any edge uv, we have $y_u'+y_v'+\sum(z_W'\colon u, v\in W\in\mathcal{Q})\geqq y_u+y_v+\sum(z_W\colon u, v\in W\in\mathcal{Q})$ so $(y', z')\in W$. Moreover $y'\geqq y$, $y'\neq y$. But

$$\begin{aligned} Z(z)-Z(z') &= z_S\cdot\{(|S|-1)/2+(|T|-1)/2\}-z_S\{(|S\setminus T|-1)/2+(|T\setminus S|-1)/2\}\\ &= z_S\cdot|S\cap T| = y'(V)-y(V),\end{aligned}$$

which contradicts Claim 1.

Therefore, $|S\cap T|$ must be odd. Define z' as follows:

$$z_W'=\begin{cases} z_W+z_S & \text{if } W=S\cup T \text{ or if } W=S\cap T \text{ and } |W|\geqq 3,\\ z_T-z_S & \text{if } W=T\\ 0 & \text{if } W=S\\ z_W & \text{if } W\in\mathcal{Q}\setminus\{S, T, S\cap T, S\cup T\}.\end{cases}$$

Again, $(y, z')\in W$ and $Z(z)=Z(z')$. Therefore (y, z') satisfies (2.7), but $\sum_{W\in\mathcal{Q}}(z_W|W|^2)<\sum_{W\in\mathcal{Q}}(z_W'|W|^2)$, a contradiction to our choice of (y, z), which establishes Claim 2.

Let $\bar{E}=\{uv\in E\colon y_u+y_v+\sum(z_S\colon S\in\mathcal{Q}\colon u, v\in S)=0\}$ and let $G^{=}=(V, \bar{E})$.

Claim 3. *Let $S\in\mathscr{Q}'$ and let $\bar{G}=G^{=}[S]\times\mathscr{Q}'[S]$. Then $G^{=}[S]$, and hence $\bar{G}$, is connected and $\bar{G}$ is nonbipartite.*

First we show that $G^{=}[S]$ is connected. If not, it has at least one component with an odd number of nodes; let K be the nodeset of this component. Let $\Delta=\min\left(\{2(y_u+y_v+\sum(z_W: W\in\mathscr{Q};\ u,v\in W)): u,v\in S,\ uv\in\delta(K)\}\cup\{z_S\}\right)$. Then $\Delta>0$. Define (y',z') by

$$y'_v=\begin{cases} y_v+\frac{1}{2}\Delta & \text{if } v\in S\setminus K\\ y_v & \text{otherwise}\end{cases}$$

$$z'_W=\begin{cases} z_W-\Delta & \text{if } W=S\\ z_W+\Delta & \text{if } W=K\\ z_W & \text{otherwise.}\end{cases}$$

Then $(y',z')\in W$ and $y'(V)-y(V)=\frac{1}{2}\Delta|S\setminus K|$ and $Z(z)-Z(z')=$ $=\Delta\left(\frac{|S|-1}{2}-\frac{|K|-1}{2}\right)=\frac{1}{2}\Delta|S\setminus K|$. But since $y'\geqq y$ and $y'\neq y$, this contradicts Claim 1.

Now suppose that $\bar{G}$ is bipartite with bipartition $V_1\cup V_2$, where $|V_1|\leqq|V_2|$. We define the following: R_1 is the set of real nodes in V_1, P_1 is the set of pseudonodes of V_1 and $\tilde{P}_1$ is the set of real nodes contained in nodes of P_1; R_2, P_2 and $\tilde{P}_2$ are defined analogously for V_2. Let

$$\begin{aligned}\Delta = \min\big(&\{z_W: W \text{ is a set of real nodes forming a node of } P_1\}\\ &\cup\{y_u+y_v+\textstyle\sum(z_W:\ W\in\mathscr{Q};\ u,v\in W):\ uv\in E(\bar{G}),\ u,v\in\tilde{P}_2\cup R_2\}\\ &\cup\{z_S\}\big).\end{aligned}$$

Define y', z' by

$$y'_v=\begin{cases} y_v+\Delta & \text{if } v\in R_1\cup\tilde{P}_1\\ y_v & \text{otherwise;}\end{cases}$$

$$z'_W=\begin{cases} z_W+\Delta & \text{if } W \text{ is the set of nodes shrunk to form a node of } P_2,\\ z_W-\Delta & \text{if } W \text{ is the set of nodes shrunk to form a node of}\\ & P_1 \text{ or if } W=S,\\ z_W & \text{otherwise.}\end{cases}$$

First, our choice of Δ ensures that $(y',z')\in W$, moreover $y'\geqq y$ and $y'(V)-y(V)=\Delta\cdot|R_1\cup\tilde{P}_1|$. But $Z(z)-Z(z')=\frac{\Delta}{2}(|\tilde{P}_1|-|P_1|-|\tilde{P}_2|+|P_2|+|S|-1)$ and since $|S|=|R_1|+|\tilde{P}_1|+|R_2|+|\tilde{P}_2|$ we have $Z(z)-Z(z')=\frac{\Delta}{2}(2|\tilde{P}_1|+2|R_1|-|R_1|-$ $-|P_1|+|P_2|+|R_2|-1)=\Delta(|\tilde{P}_1|+|R_1|)+\frac{\Delta}{2}(|V_2|-|V_1|-1)$. Since $|S|$ is odd and

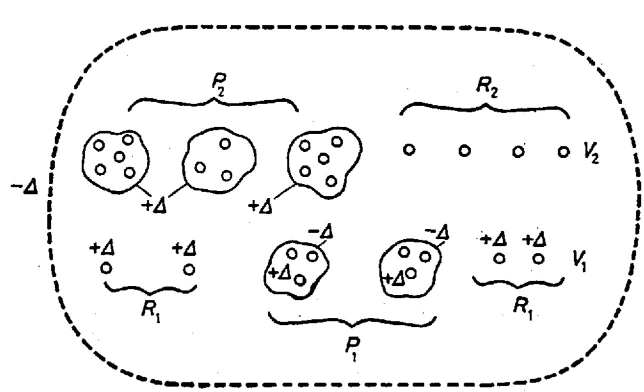

Fig. 2

$|V_1| \leqq |V_2|$, we must have $|V_2|-|V_1|-1 \geqq 0$, so $Z(z)-Z(z') \geqq y(V)-y(V)$ which contradicts Claim 1, since $V_1 \neq \emptyset$ implies $y' \neq y$.

Claim 4. *For all distinct $S, T \in \mathscr{Q}'$, $S \cap T = \emptyset$. For any $S \in \mathscr{Q}'$, for all $v \in S$, $y_v = -\frac{1}{2} z_S$.*

For any $S \in \mathscr{Q}'$, we let $\varrho_S = \sum (z_W \colon W \in \mathscr{Q}', W \supseteq S)$. Let S be a minimal member of $\mathscr{Q}'$. By Claim 3, $G^=[S]$ is nonbipartite and connected. By considering the nodes belonging to an odd cycle of $G^=[S]$ we see that we must have $y_v = -\frac{1}{2}\varrho_S$ for all nodes v of this cycle, and so, since $G^=[S]$ is connected,

$$y_v = -\frac{1}{2}\varrho_S \quad \text{for all} \quad v \in S. \tag{2.8}$$

If there are nondisjoint members of $\mathscr{Q}'$, then since it is a nested family, we can choose a set $T \in \mathscr{Q}'$ which is not minimal in $\mathscr{Q}'$, but all members of $\mathscr{Q}'$ contained in T are minimal. Let S be a member of $\mathscr{Q}'$ properly contained in T. Let $v \in S$.

By (2.8), we have $y_v = -\frac{1}{2}(\varrho_T + z_S)$. Now consider the graph $\bar{G} = G[T] \times \mathscr{Q}'[T]$. By (2.8), for any nodes u, w belonging to the same pseudonode W of $\bar{G}$, we have $y_u = y_w = -\frac{1}{2}\varrho_W$. Since $\bar{G}$ is connected, it is an easy inductive exercise to show that, for each node $u \in T$, we either have $y_u = -\frac{1}{2}(\varrho_T + z_S)$ if u or the pseudonode containing it is at an even distance from S in $\bar{G}$ or $y_u = -\frac{1}{2}(\varrho_T - z_S)$ if this distance is odd. Moreover, each edge of $\bar{G}$ joins nodes having different values. But this then implies that $\bar{G}$ is bipartite, which contradicts Claim 3. Hence all members of $\mathscr{Q}'$ are disjoint, which together with (2.8) establishes the claim. (Note that if G is bipartite, then $\mathscr{Q} = \mathscr{Q}' = \emptyset$ and so Claims 2, 3 and 4 are vacuous.)

Now it is easy to complete the proof of Theorem 2.6. Let V^+, V^- and V^0 be the sets of nodes v where $y_v>0$, $y_v<0$ and $y_v=0$, respectively.

Since $z\geqq 0$ we see the following:

(i) No edge $uv\in E$ can join two nodes of V^- unless they belong to the same $S\in\mathscr{Q}$ and $z_S>0$, (or else we would contradict feasibility).

(ii) $\Gamma(V^-)\subseteqq V^+$.

(iii) $E^=\subseteqq\bigcup\{\gamma(S):\ S\in\mathscr{Q}'\}\cup\{uv:\ u\in V^+,\ v\in V^-\}\cup\gamma(V^0)$.

But now if we let $X=V^-$ and $\alpha=a_0$ and consider the vectors $y^{X,\alpha}$ and $z^{X,\alpha}$, we see that they give a member of W for which we have equality in (iii) above. But since (y, z) generates an extreme ray of W, the set of inequalities defining W which hold as equations must be maximal, so we must have had, in fact, $y=y^{X,\alpha}$ and $z=z^{X,\alpha}$ and the proof is complete. ▮

We can now combine Proposition 2.5 and Theorem 2.6 to obtain the following system sufficient to define $PMS(G)$ for a general graph $G=(V, E)$. For any $S\subseteqq V$, we let $\theta(S)$ be the number of connected components of $G[S]$.

Theorem 2.7. *For any graph* $G=(V, E)$,

$$PMS(G) = \{x\in R^V:$$

$$0\leqq x\leqq 1$$

$$x(S)-x(\Gamma(S))\leqq |S|-\theta(S) \tag{2.9}$$

for all $S\subseteqq V$ *such that every component of* $G[S]$ *consists of a single node or else is a nonbipartite graph with an odd number of nodes*$\}$.

We conclude this section with two remarks. First, it is not true that every extreme ray of W has the form $(y^{X,\alpha}, z^{X,\alpha})$ for some $X\in\mathscr{T}$ and $\alpha\geqq 0$. Consider the graph of Figure 3.

We let $\hat{y}_v=-1$ for all nodes $v\notin\{b,f\}$, and let $\hat{y}_b=\hat{y}_f=1$. We let $\hat{y}_S=0$ for all $S\in\mathscr{Q}\setminus\{\{c, g, d\}, \{g, d, e\}\}$ and define $\hat{y}_S=2$ for these two triangles. Note that we have equality in the constraints defining W for every edge except gd. It is easy to check that $(\hat{y}, \hat{z})$ is the unique member of W, up to nonnegative multiples,

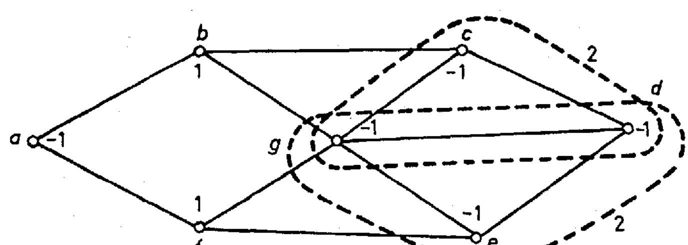

Fig. 3

which satisfies this and has $\hat{y}_S=0$ for all $S\in\mathcal{D}\setminus\{\{c,g,d\},\{g,d,e\}\}$ and so generates an extreme ray of W. However, the valid inequality for $PMS(G)$ obtained from $(\hat{y},\hat{z})$ by Proposition 2.5 is

$$x_a+x_c+x_d+x_g+x_e-x_b-x_f \leqq 4. \tag{2.10}$$

If we let $S=\{c,g,d,a\}$, then we obtain the following inequality from Theorem 2.7:

$$x_a+x_c+x_g+x_d-x_b-x_f-x_e \leqq 2.$$

If we add twice the valid inequality $x_e\leqq 1$ to this, we obtain (2.10), so this is an example of an extreme ray of W generating a valid, but non-facet-inducing inequality for $PMS(G)$.

Second, we note that it is easy to deduce Tutte's theorem [9] characterizing those graphs which have perfect matchings from Theorem 2.7. For Theorem 2.7 implies that G has a perfect matching if and only if the vector $\hat{x}$ obtained by defining $\hat{x}_v=1$ for all $v\in V$ satisfies our linear system. But this holds if and only if $|\Gamma(S)|\geqq\theta(S)$ for all $S\in\mathcal{T}$. So if G has no perfect matching, then there exists a set $X\subseteqq V$ such that $G\setminus X$ has more than $|X|$ odd components — which is the "hard" direction of Tutte's theorem.

3. Relationship to the Bipartite Case

In [2] we showed that the following linear system is sufficient to define $PMS(G)$ for a bipartite graph $G=(V_1\cup V_2, E)$:

$$\begin{aligned} PMS(G) = \{x\in\mathbf{R}^{V_1\cup V_2}:& \\ &0\leqq x\leqq 1,\\ &x(S)-x(\Gamma(S))\leqq 0 \quad \textit{for all} \quad S\subseteqq V_1,\\ &x(V_1)-x(V_2)=0\}. \end{aligned}$$

We can deduce this result easily from Theorem 2.7. Applying Theorem 2.7 to G, we obtain an inequality (2.10) for every $S\subseteqq V$ such that S is independent, i.e., no two members are adjacent. This inequality will be $x(S)-x(\Gamma(S))\leqq 0$. Combining the inequalities corresponding to V_1 and V_2 we obtain the equation $x(V_1)-x(V_2)=0$. With this equation, it is straightforward to deduce the inequality $x(S)-x(\Gamma(S))\leqq 0$ for independent sets $S\not\subseteqq V_1$ from those corresponding to $S\subseteqq V_1$. (See [2] for details.)

What is more surprising is that we can deduce Theorem 2.7 from the bipartite result, plus the so-called Edmonds—Gallai structure theorem. (Anderson [1] used an argument with a similar structure to derive Tutte's Theorem from Hall's Theorem, which characterizes those bipartite graphs having perfect matchings.)

The derivation of the nonbipartite result is easier if we use the following minor extension of the bipartite theorem. Let $G=(V_1\cup V_2, E)$ be a bipartite graph. We say that $W\subseteqq V_1\cup V_2$ is *V_1-matchable* if there is a matching of $G[W]$ which saturates all nodes of $W\cap V_1$. In other words, W consists of a set U such that $G[U]$ has a perfect matching plus, possibly, some additional nodes of V_2.

Theorem 3.1. *For a bipartite graph $G=(V_1\cup V_2, E)$, the convex hull of the incidence vectors of the V_1-matchable subsets of $V_1\cup V_2$ is given by*

$$0 \leqq x \leqq 1$$

$$x(S)-x(\Gamma(S)) \leqq 0 \quad \text{for all} \quad S \subseteq V_1.$$

Note that the only change from the defining linear system for $PMS(G)$ is that the equation has been removed, leaving only the inequality $x(V_1)-x(V_2)\leqq 0$.

This result is derived in [2] as a special case of lattice polyhedra. It can also be easily deduced using the projection method of [2], or, it can be deduced directly from the characterization of $PMS(G)$. (See [3].)

A graph $G=(V, E)$ is called *critical, (or hypomatchable)* if, for every $v\in V$, $G\setminus\{v\}$ has a perfect matching. A matching which saturates all nodes but one of G is called a *near perfect* matching. A critical graph is nonbipartite, and has an odd number of nodes. The *Edmonds—Gallai partition* of a graph $G=(V, E)$ is the partition of V into $O(G)\cup I(G)\cup P(G)$ defined by

$O(G) = \{v\in V$: some maximum matching of G leaves v unsaturated$\}$;
$I(G) = \Gamma(O(G))$;
$P(G) = V\setminus(I(G)\cup O(G))$.

Note that every maximum matching saturates all nodes of $I(G)\cup P(G)$, and if G has a perfect matching then $P(G)=V$ and $I(G)=O(G)=\emptyset$.

Theorem 3.2. (Edmonds—Gallai Theorem, see Lovász and Plummer [7] §3.2). *For any graph G,*
(i) *every component of $O(G)$ is critical;*
(ii) *a matching M is maximum if and only if*
 a) *M induces a perfect matching of $G[P(G)]$;*
 b) *each node in $I(G)$ is joined by an edge of M ot a node of a distinct component of $G[O(G)]$;*
 c) *M induces a near perfect matching on each component of $G[O(G)]$.*

If Edmonds' maximum matching algorithm [6] is applied to G, is determines the Edmonds—Gallai partition in polynomial time.

Let $c=(c_v\colon v\in V)$ be a vector of node costs. We consider here the linear program

$$\text{maximize } cx$$

$$\text{subject to} \quad 0\leqq x\leqq 1,$$

$$x(S)-x(\Gamma(S)) \leqq |S|-\theta(S) \quad \text{for} \quad S\in\mathcal{T}. \tag{3.1}$$

(Recall that $\mathcal{T}=\{S\subseteq V$: every component of $G[S]$ has an odd number of nodes$\}$.)

The dual linear program is the following:

$$\text{minimize} \quad y(V)+\sum(z_S\cdot(|S|+\theta(S))\colon S\in\mathcal{T})$$

$$\text{subject to} \quad y, z\geqq 0,$$

$$y_v+\sum(z_S\colon S\in\mathcal{T}, v\in S)-\sum(z_S\colon S\in\mathcal{T}, v\in\Gamma(S))\geqq c_v \quad \text{for all } v\in V. \tag{3.2}$$

We will show that for any vector c of node costs these linear programs have feasible solutions x^* and y^*, z^*, giving identical objective values and such that x^* is $0-1$ valued. Since the $0-1$ valued solutions to (3.1) are precisely the incidence vectors of members of $\mathscr{W}$, this will show that $PMS(G)=\{x\in\mathbf{R}^V\colon 0\leqq x\leqq 1$ and $x(S)-x(\Gamma(S))\leqq|S|-\theta(S)$ for all $S\in\mathscr{T}\}$. Then all we need to show to obtain Theorem 2.7 is that constraints (3.1) for $S\in\mathscr{T}$ such that $G[S]$ has nonsingleton bipartite components are redundant. But this is easy for suppose K is such a component of $G[S]$, for $S\in\mathscr{T}$. Let K_1 and K_2 be the nodesets of the two parts of K, where $|K_1|>|K_2|$ and let $S'=S\setminus K_2$. (Each node of K_1 is a singleton in $G[S']$.) The constraint (3.1) corresponding to S is implied by the sum of the constraint (3.1) corresponding to S', plus twice the sum of the constraints $x_v<1$ for $v\in K_2$, plus the sum of the constraints $-x_v\leqq 0$ for $v\in\Gamma(S)\setminus(S')$.

We obtain y^*, z^* and x^* as follows. (See [3] for more details.) First, we consider the graph G' induced by the set W of nodes with nonnegative costs. If G' has a perfect matching M which saturates all nodes with strictly positive costs, then let x^* be the incidence vector of M. Let $y_v^*=\max\{0, c_v\}$ for all $v\in V$ and let $z^*=O$. These vectors are feasible and optimal.

If no such M exists, then we construct a bipartite graph $\bar{G}$ based on the Edmonds—Gallai partition of $G'=G[W]$. We let $\bar{G}$ be the graph obtained from $G[O(G')\cup\Gamma(O(G'))]$ by shrinking all components of $O(G')$ to from pseudonodes and deleting any edges with both ends in $\Gamma(O(G'))$. (Note that $\Gamma(O(G'))$ is the neighbour set of $O(G')$ in G, not G'.) Let V_1 be the set of nodes of $\bar{G}$ corresponding to $O(G')$.

For each pseudonode v, we define c_v to the minimum cost of the nodes shrunk to form the pseudonode. For each real node, the cost remains as in G. We now find a V_1-matchable subset $X^*$of $\bar{G}$, for which $c(X^*)$ is maximum. Then it can be seen that any perfect matching of $G[X^*]$ can be extended to a matching of G such that, if x^* is the incidence vector of the set of saturated nodes, then $cx^*=c(P(G'))+$ $+c(O(G'))+c(X^*)$.

Let $\bar{y}$, $\bar{z}$ be an optimal dual solution to the linear program of maximizing cx, subject to the constraints of Theorem 3.1 for $\bar{G}$. We obtain the required y^*, z^* as follows. If $v\in(G')$, we let $y_v^*=0$. If $v\in\Gamma(O(G'))$ we let $y_v^*=\bar{y}_v$. If v belongs to a pseudonode K of $\bar{G}$, then we let $y_v^*=\bar{y}_K+c_v-c_{\bar{v}}$, where $\bar{v}$ is a node of K for which $c_{\bar{v}}$ is minimum. All other nodes have $y_v^*=0$. We define z_T^* to be equal to the value of $\bar{z}_S$, where T is the set of nodes of G contained in the nodes and pseudonodes belonging to S, for all $S\subseteqq V_1$. Otherwise, $z_T^*=0$.

It can be verified that y^*; z^* is a feasible dual solution, and gives an objective value equal to cx^*, proving optimality. Again, see [3] for details.

This construction shows that if c is integer valued, then there exists an optimal dual solution which is integer valued, i.e., the system (3.1) is *totally dual integral.* For if c is integer valued, so too will be the costs defined for the nodes of $\bar{G}$. We showed in [2] that the system of Theorem 3.1 was totally dual integral, so $\bar{y}$ and $\bar{z}$ can be chosen to be integer valued, which will cause y^* and z^* to also be integer valued.

4. Facets of $PMS(G)$

In this section we characterize those inequalities which induce facets of $PMS(G)$. For the trivial inequalities the situation is particularly simple. If G is nonbipartite, then the inequality $x_v\geqq 0$ is facet inducing unless v is adjacent to a degree one node w, in which case the inequality is obtained by adding the inequality $-x_w\leqq 0$ to the inequality (2.9), taking $S=\{w\}$. The inequality $x_w\leqq 1$ is facet inducing unless w is a degree one node (in which case it is implied by the inequality (2.9) with $S=\{w\}$ plus $x_v \leqq 1$, where v is the neighbour) or $G=(V, E)$ is a triangle (when it is obtained by adding the inequalities (2.9) for $S=V$ and $S=\{w\}$). The proofs are easy and we leave the details to the reader. The bipartite case is treated in [2].

The main interest is in characterizing those inequalities (2.9) which induce facets, which we do for general (bipartite or nonbipartite) graphs. We make use of two lemmas. The first follows easily from Tutte's theorem, we give its proof for the sake of completeness.

Lemma 4.1 (cf. Pulleyblank and Edmonds [8]). *If $G=(V, E)$ is not critical but $|V|$ is odd, then there exists $X \subseteqq V$ such that every component of $G[V \setminus X]$ is critical, there are at least $|X|+1$ such components and every node in X is adjacent to a node in $V \setminus X$.*

Proof. We use induction of the size of G. If G is not critical, then there exists $v \in V$ such that $G \setminus \{v\}$ has no perfect matching. By Tutte's theorem, there exists $X' \subseteqq V \setminus \{v\}$ such that $G[V \setminus (X' \cup \{v\})]$ has at least $|X'|+2$ odd components. Choose such an X' for which the number of odd components of $G'=G[V \setminus (X' \cup \{v\})]$ is maximum. If $\Gamma(\{v\}) \subseteqq X'$, then let $X=X'$, otherwise, let $X=X' \cup \{v\}$. In either case, G' has at least $|X|+1$ odd components. If a component K of G' had an even number of nodes, then adding an arbitrary node of K to X' would contradict the maximality property of G'. If any node of X' is adjacent only to nodes of X' then we can remove this node and again contradict the maximality of G'. Finally, if some odd components K of G' is not critical, then by induction there exists $\bar{X} \subseteqq V(K)$ satisfying the conditions of the lemma. Again, $X' \cup \bar{X}$ contradicts the maximality property of G'. ∎

The second lemma characterizes those sets $T \in \mathscr{W}$ which satisfy (2.9) with equality for a given $S \in \mathscr{T}$. (Recall that $\mathscr{W}$ is the family of subsets of nodes saturated by some matching of G.)

Lemma 4.2. *Let $S \in \mathscr{T}$ and let T be the set of nodes saturated by some matching of G. Then the incidence vector $\hat{x}$ of T satisfies*

$$\hat{x}(S)-\hat{x}(\Gamma(S)) = |S|-\theta(S)$$

if and only if

(4.1) *for each component K of $G[S]$, T contains all but possibly one node of K,*

(4.2) *there exists a perfect matching of $G[T]$ which joins each node of $T \cap \Gamma(S)$ to a node of a distinct component K of $G[S]$ for which $V(K) \subseteqq T$.*

Proof. Let M be a perfect matching of $G[T]$. For each component K of $G[S]$, let M_K be the set of edges of M with both ends in K. Let M_1 be the set of edges of M which join nodes of S to nodes of $\Gamma(S)$ and let M_2 be the set of edges of M which join nodes of $\Gamma(S)$ to nodes not in S. For each component K of $G[S]$, $2|M_K| \leqq |V(K)|-1$, so if $\hat{x}$ is the incidence vector of T, then

$$\begin{aligned}\hat{x}(S)-x(\Gamma(S)) &\leqq \sum (2|M_K|: K \text{ is a component of } G[S]) \\ &\quad +|M_1|-|M_1|-|M_2| \\ &\leqq \sum (|V(K)|-1: K \text{ is a component of } G[S]) \\ &= |S|-\theta(S).\end{aligned}$$

Therefore we have equality if and only if $M_2=\emptyset$ and $2|M_K|=|V(K)|-1$ for every component K of $G[S]$, i.e., if and only if (4.1) and (4.2) hold. ▮

Theorem 4.3. *Let G be nonbipartite. For $S\in\mathcal{T}$, the inequality (2.9) is facet inducing for $PMS(G)$ if and only if*

(4.3) *every component of $G[S]$ is critical;*

(4.4) *every component of $G\setminus(S\cup\Gamma(S))$ is nonbipartite;*

(4.5) *the graph obtained from $G[S\cup\Gamma(S)]$ by deleting all edges with both ends in $\Gamma(S)$ is connected.*

Proof. We first show the necessity of our conditions. If (4.5) is violated, then the inequality (2.9) corresponding to S can be deduced by adding the inequalities corresponding to $S\cap V(K)$ for all components K of $G[S\cup\Gamma(S)]$.

Suppose (4.4) is violated and $G[V\setminus(S\cup\Gamma(S))]$ has a bipartite component K. Let K_1 and K_2 be the nodesets of the parts. Adding the inequalities (2.9) corresponding to $S\cup K_1$ and $S\cup K_2$ gives us exactly twice the inequality (2.9), so the inequality is redundant.

Suppose (4.3) is violated. If a component K of $G[S]$ is not critical then we apply Lemma 4.1 to find $\bar{X}\subseteq V(K)$ such that every component of $K\setminus\bar{X}$ is critical, there are at least $|\bar{X}|+1$ such components and $\Gamma(V(K)\setminus\bar{X})=\bar{X}$. Let $S'=S\setminus\bar{X}$. Then $\Gamma(S')\subseteq\Gamma(S)\cup\bar{X}$ and $\theta(S')>\theta(S)-1+|\bar{X}|$, i.e., $\theta(S')\geq\theta(S)+|\bar{X}|$. To the inequality (2.9) corresponding to $\bar{S}$, we add twice the inequality $x_v\leq 1$ for all $v\in\bar{X}$. This yields an inequality which implies $x(S)-x(\Gamma(S))\leq|S'|-\theta(S')+2|\bar{X}|\leq$ $\leq|S|-\theta(S)$. Hence the inequality (2.9) corresponding to S was redundant.

Now we prove the sufficiency. Suppose that (4.3)—(4.5) hold. We show that the inequality (2.9) is facet inducing by showing that for each other inequality $ax\leq\alpha$ used to define $PMS(G)$, we can find $\hat{x}\in PMS(G)$ satisfying $a\hat{x}<\alpha$ but $\hat{x}(S)-\hat{x}(\Gamma(S))=|S|-\theta(S)$. For then if we take a positive convex combination of these points, we obtain $x^*\in PMS(G)$ for which the only tight inequality is (2.9). For $\varepsilon>0$, $(1+\varepsilon)x^*$ violates (2.9), so this point is not in $PMS(G)$. But for ε sufficiently small, this is the only violated inequality, so it is facet inducing.

By (4.3), for each component K of $G[S]$ we can choose an arbitrary node v_K of K and construct a perfect matching of $K\setminus\{v_K\}$. If we do this for all components, the set $\bar{T}$ of saturated nodes satisfies (4.1) and (4.2) so the incidence vector $x^{\bar{T}}$ satisfies (2.9) with equality. Now we consider the three types of inequalities:

Case 1. $x_v\leq 1$ for $v\in V$. For any node v, by choosing an appropriate $\bar{T}$ as above we have $x_v^{\bar{T}}=0$, i.e., $x_v^{\bar{T}}<1$, as required.

Case 2. $x_v\geq 0$ for $v\in V$. Choose $\bar{T}$ as above such that for each component K of $G[S]$, the node of K not in $\bar{T}$ is adjacent to a node of $\Gamma(S)$. For $v\in S$, if $v\in\bar{T}$ then let $T=\bar{T}$. If $v\in S\setminus\bar{T}$, then let u be an adjacent node of v in $\Gamma(S)$ and let $T=\bar{T}\cup\{u,v\}$. If $v\in\Gamma(S)$, then let w be a node of $S\setminus\bar{T}$ in a component of $G[S]$ containing a node adjacent to v and let $T=\bar{T}\cup\{v,w\}$. Finally, if $v\in V\setminus(S\cup\Gamma(S))$ then by (4.4) there exists $w\in V\setminus(S\cup\Gamma(S))$ such that v and w are adjacent. Let $T=\bar{T}\cup\{v,w\}$. In every case, there exists a perfect matching of $G[T]$ and the incidence vector x^T satisfies $x_v^T>0$ and $x^T(S)-x^T(\Gamma(S))=|S|-\theta(S)$.

Case 3. $x(U)-\theta(\Gamma(U)) \leqq |U|-\varkappa(U)$ for some $U \in \mathcal{T} \setminus \{S\}$. Suppose that every $\hat{x} \in PMS(G)$ which satisfies (2.9) with equality also satisfies $\hat{x}(U)-\hat{x}(\Gamma(U)) = |U|-\theta(U)$. If $G[S]$ has any component K with more than one node, then by considering $\bar{T}$ as above which leaves each node of K in turn unsaturated, we see that either $V(K) \subseteq U$, $V(K) \subseteq \Gamma(U)$ or $V(K) \cap (U \cup \Gamma(U)) = \emptyset$. Suppose that some component K of $G[U]$ having three or more nodes were not contained in S. We could take any $\bar{T}$ as above for S, and its incidence vector $\hat{x}$ would satisfy $\hat{x}(U) - \hat{x}(\Gamma(U)) < |U| - \theta(U)$, by Lemma 4.2. Therefore

(4.6) every nontrivial component of $G[U]$ is contained in $G[S]$.

Suppose $W = U \setminus S \neq \emptyset$. By (4.6), W is an independent set of nodes. If any nodes of $\Gamma(W) \setminus \Gamma(S)$ were adjacent, or adjacent to a node not in $\Gamma(S) \cup W$, we could start with any $\bar{T}$ as above for S, then add such an adjacent pair of nodes and the incidence vector $\hat{x}$ would satisfy (2.9) for S, but not for U. Therefore $G[W \cup (\Gamma(W) \setminus \Gamma(S))]$ is a bipartite component (or a collection of such components) of $G \setminus (S \cup \Gamma(S))$, which contradicts (4.4). Therefore

(4.7) $U \subseteq S$, and hence $\Gamma(U) \subseteq \Gamma(S)$.

Finally, suppose there exists $w \in S \setminus U$. Choose such a w adjacent to a node u of $\Gamma(U)$, which is possible by (4.5). Then if we take $\bar{T}$, as above, together with u and w, the incidence vector again satisfies (2.9) for U but not S as required. ∎

For a case of a bipartite graph $G = (V_1 \cup V_2, E)$, we showed in [2] that, for any $S \subseteq V_1$, the inequality $x(S) - x(\Gamma(S)) \leqq 0$ was facet inducing if and only if both $G[S \cup \Gamma(S)]$ and $G[(V_1 \setminus S) \cup (V_2 \setminus \Gamma(S))]$ were connected. Since, in the bipartite case, every $x \in PMS(G)$ satisfies the equation $x(V_1) - x(V_2) = 0$, we see that any facet is induced by several different inequalities of the form (2.9). In particular, suppose that S is some subet of $V_1 \cup V_2$; let $S_1 = S \cap V_1$ and $S_2 = S \cap V_2$. Then (4.3) holds if and only if no edge joins two nodes of S and (4.4) holds if and only if every node is in $S \cup \Gamma(S)$. If either $G[S_1 \cup \Gamma(S_1)]$ or $G[S_2 \cup \Gamma(S_2)]$ were not connected, then (4.5) would be violated. However, connectivity of G requires there be edges present joining nodes of $\Gamma(S_1)$ to nodes of $\Gamma(S_2)$ and if we delete them, then the graph is no longer connected. Thus it is true that for a bipartite graph G, every facet of $PMS(G)$ induced by an inequality (2.9), is induced by such an inequality for S such that no edge joins two nodes of S, every node belongs to $S \cup \Gamma(S)$ and the graph obtained by deleting all edges with both ends in $\Gamma(S)$ has exactly two components. For by adding the equation $x(V_1) - x(V_2) = 0$ to such an inequality we obtain $x(S_1) = x(\Gamma(S_1)) \leqq 0$ for $S_1 \subseteq V_1$, satisfying the conditions in [2]. In other words, Theorem 4.2 is also valid for bipartite graphs. It is also easy to modify the proof of this theorem to obtain this directly.

5. Conclusions

In [2] we introduced a technique for obtaining a linear system sufficient to define a combinatorial polyhedron P from a defining linear system for a larger polyhedron Q such that P is a projection of Q. In this paper we give another, more complex, application of this method. The method consists of finding a set of generators for a particular cone and then "post multiplying" the generators to obtain the defining inequalities. In the case of perfectly matchable subgraph polyhedra of general graphs, we did not describe a complete set of generators of the relevant cone. However we did describe a set of generators sufficient to produce all facet inducing inequalities. Thus one important point illustrated here is that it is not essential to have a complete set of generators of the cone, in order to obtain the desired projection.

We also discussed the relationship of the nonbipartite result to the earlier bipartite theorem [2]. In particular we showed that the bipartite theorem, plus the Edmonds—Gallai structure theorem are sufficient to deduce the nonbipartite result.

An interesting related problem is the so-called separation problem for $PMS(G)$: Given a vector $\hat{x}\in\mathbf{R}^V$, either show that $\hat{x}\in PMS(G)$ (by providing a set of vertices of $PMS(G)$, of which it is a convex combination) or else show that it is not, by giving an inequality $ax\leqq\alpha$ valid for all $x\in PMS(G)$, but such that $a\hat{x}>\alpha$. This problem was solved by W. H. Cunningham and J. Green—Krotki [4] as a special case of the problem of determining whether there exists a (usually fractional) vector x belonging to the matching polyhedron $M(G)$ such that $x(\delta(v))$ lies between prescribed bounds, for all nodes v. Their results also provide another proof of Theorem 2.7.

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